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Semi-Analytical Static Nonlinear Structural Sensitivity Analysis

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The paper discusses two uses of the semi-analytical method for nonlinear sensitivity analysis. First, the application to noncritical response is analyzed. It is shown that the semi-analytical method is equivalent to a particular form of the overall finite difference approach. It is concluded that the difference between overall finite differences and the semi-analytical approach is blurred for the case of nonlinear static response. Next, the application of the semi-analytical method to calculating sensitivities of limit loads is discussed. A formulation that is easy to implement in general purpose finite element programs is derived. Three examples are used to demonstrate the application of the formulation and to explore its accuracy.

Introduction

IN the past few years there has been much interest in sensitivity of nonlinear static structural response (e.g., Ref. 1). Most of the work has been concerned with analytical calculation of the sensitivity derivatives, with very little interest in finite difference and semi-analytical sensitivity calculations. However, in terms of implementation with general purpose finite element programs, these last two methods are easier to implement than analytical approaches.

Finite difference calculations are usually much more expensive than analytical derivative calculations. However, for nonlinear analysis, this cost handicap may not apply. Indeed, if Newton's method is used for the analysis, then the cost of analyzing a perturbed design may be a small fraction of the cost of the nonlinear analysis of the nominal design. Furthermore, Ref. 2 shows that accuracy problems associated with the finite difference approach can be circumvented by adding the force residual left in the (not fully converged) nominal design to the perturbed design. The first part of the present paper shows that with the approach of Ref. 2 the distinction between the semi-analytical and finite difference methods becomes blurred.

The second part of the paper deals with sensitivity of limit loads. Analytical procedures for derivatives of limit loads have been derived by Wu and Arora[3] and Komarakul-na-nakorn and Arora.[4] The paper presents the semi-analytical method for calculating the sensitivity of limit loads and shows the method can be easily implemented with general purpose finite element programs. Three examples are used to demonstrate the calculation of semi-analytical derivatives of limit load.

Analysis and Sensitivity of Noncritical Response

After discretization, the equation of equilibrium for static nonlinear response may be written as

$$f(u, x) = \mu p(x) \tag{1}$$

where f is the vector of internal forces, which is a function of the displacement vector u and a design variable x, p is the vector of applied loads, and μ is an amplitude parameter. It is assumed here that the applied loads are not a function of the displacement field (that is, they are dead loads), but accounting for live loads is not difficult.

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Consider first the solution of the equations of equilibrium for a fixed value of the design variable, so that the dependence on x can be temporarily omitted. The equations of equilibrium are typically solved for a series of μ values by Newton's method or the modified Newton method. That is, given an estimate u_0 for u, a better estimate u_1 is found by using

$$f(u_1) \approx f(u_0) + J(u_1 - u_0) = \mu p \tag{2}$$

where J is the Jacobian $\partial f/\partial u$ also known as the tangential stiffness matrix. Then

$$u_1 = u_0 + J^{-1}[\mu p - f(u_0)] \equiv J^{-1} r_0 \tag{3}$$

where r_0 is the residual of the equations of equilibrium. The iteration then proceeds by replacing u_0 by u_1 and repeating until the magnitude of the residual satisfies a given convergence tolerance. To obtain a highly nonlinear response, it may be necessary to use many intermediate values of the amplitude parameter μ to insure the convergence of Newton's method. The solution for one value of μ then becomes the initial estimate for the next value of μ. This process can be two orders of magnitude more expensive than a linear analysis because of the need for repeated calculation of J and f.

The derivative of u with respect to the design variable x may be obtained by differentiating Eq. (1) to obtain

$$J\frac{du}{dx} = \mu\frac{dp}{dx} - \frac{\partial f}{\partial x} \tag{4}$$

The solution of Eq. (4) for the derivative of the response is obviously much cheaper than the calculation of the response itself in that it does not require any iterations at all. The cost differential becomes even more pronounced when we need derivatives with respect to several design variables because only the right-hand side of Eq. (4) changes when we change design variables. However, in many cases programming the right-hand side of Eq. (4) is tedious, cumbersome, or impossible because of problems of access to the source code of the structural analysis package or insufficient documentation as to how f is generated. In such cases, a semi-analytical approach is favored. In the semi-analytical approach, the right-hand side of Eq. (4) is calculated by finite differences. That is

$$\frac{du}{dx} \approx -J^{-1}\left(\mu\frac{\Delta p}{\Delta x} - \frac{\Delta f}{\Delta x}\right) \tag{5}$$

where $\Delta/\Delta x$ denotes a finite difference operator. Usually first-order forward differences are used, that is,

$$\frac{\Delta f}{\Delta x} = \frac{f(u, x + \Delta x) - f(u, x)}{\Delta x} \tag{6}$$

However, the semi-analytical method is known to have accuracy problems for shape design variables in beam, plate, or shell problems (e.g., Ref. 5). Such problems are usually overcome by using higher order difference formulas such as central differences. Even though this may double or triple the cost of the sensitivity calculation, this cost still remains a small fraction of the cost of the analysis.

The other option for calculating response derivatives is an overall finite difference approach. Again, most often first-order forward differences are used, so that the equations of equilibrium are perturbed as

$$f(u_\Delta, x + \Delta x) = \mu p(x + \Delta x) \tag{7}$$

and then the derivative is approximated as

$$\frac{\mathrm{d}u}{\mathrm{d}x} \approx \frac{u_\Delta - u}{\Delta x} \tag{8}$$

Because a nonlinear analysis is so much more expensive than analytical or semi-analytical sensitivity calculation, it may appear unreasonable to repeat the analysis [as Eq. (7) implies] to obtain the overall finite difference derivative. However, the cost of solving Eq. (7) need not be as high as the cost of solving Eq. (1) because we can use u as an initial estimate for u_Δ. Since Δx is small, we can expect convergence to u_Δ in one or two iterations. A difficulty associated with using u as an initial estimate to u_Δ is that u obtained from the solution of Eq. (1) is rarely fully converged to machine precision. Therefore, when Eq. (7) is solved by iterating from u, two processes take place simultaneously. The first is the change in the solution due to the change from x to $x + \Delta x$, and the second is further refinement of the solution at x. The second effect can be larger than the first one for small values of Δx, thus destroying the accuracy of the finite difference derivative. Reference 2 (p. 259) suggests that this problem can be overcome by subtracting the residual left in the solution of Eq. (1) from Eq. (7). That is, if $\tilde{u}$ is the approximation to u obtained by Newton's method for Eq. (1), then the residual is

$$r(\tilde{u}) = \mu p(x) - f(\tilde{u}, x) \tag{9}$$

and we replace Eq. (7) by

$$f(u_\Delta, x + \Delta x) = \mu p(x + \Delta x) - r(\tilde{u}) \tag{10}$$

With Eq. (10) used to calculate u_Δ, starting from $\tilde{u}$ as initial guess, the cost and accuracy of the overall finite difference method would be acceptable. Interestingly, this overall finite difference procedure is essentially equivalent to the semi-analytical method. Indeed, if we performed only one iteration of Newton's method on Eq. (10), then we would get $\tilde{u}_\Delta$ as an approximation to u_Δ, where

$$\begin{aligned}\tilde{u}_\Delta &= \tilde{u} + J^{-1}[\mu p(x + \Delta x) - f(\tilde{u}, x + \Delta x) - r(\tilde{u})] \\ &= \tilde{u} + J^{-1}[\mu p(x + \Delta x) - \mu p(x) - f(\tilde{u}, x + \Delta x) + f(\tilde{u}, x)]\end{aligned} \tag{11}$$

Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} \approx \frac{\tilde{u}_\Delta - \tilde{u}}{\Delta x} = -J^{-1}\left(\mu\frac{\Delta p}{\Delta x} - \frac{\Delta f}{\Delta x}\right) \tag{12}$$

which is identical to the semi-analytical result, Eq. (5). We conclude that the semi-analytical derivative could also be viewed as the overall finite difference derivative based on a single Newton iteration for the perturbed solution! Unlike the linear case, the difference between the overall finite difference derivative and the semi-analytical derivative is blurred.

Analysis and Sensitivity of Limit Loads

To be able to follow the load-displacement curve past limit points, it is usual to follow a path parameter other than the load, such as a displacement component known to increase monotonically or an arc length measure. Denoting such a parameter by α and differentiating Eq. (1) with respect to α for a given value of x, we get

$$Ju' = \mu' p \tag{13}$$

where a prime denotes a derivative with respect to α. At a limit point $\mu' = 0$, which indicates that the tangential stiffness matrix J is singular with u' being then a multiple of the eigenvector corresponding to the zero eigenvalue (assumed here to be simple). It is possible to locate the limit load from the condition that J is singular there, by using the equation

$$\left[J + \frac{\mathrm{d}J}{\mathrm{d}\mu}\Delta\mu\right]u' = 0 \tag{14}$$

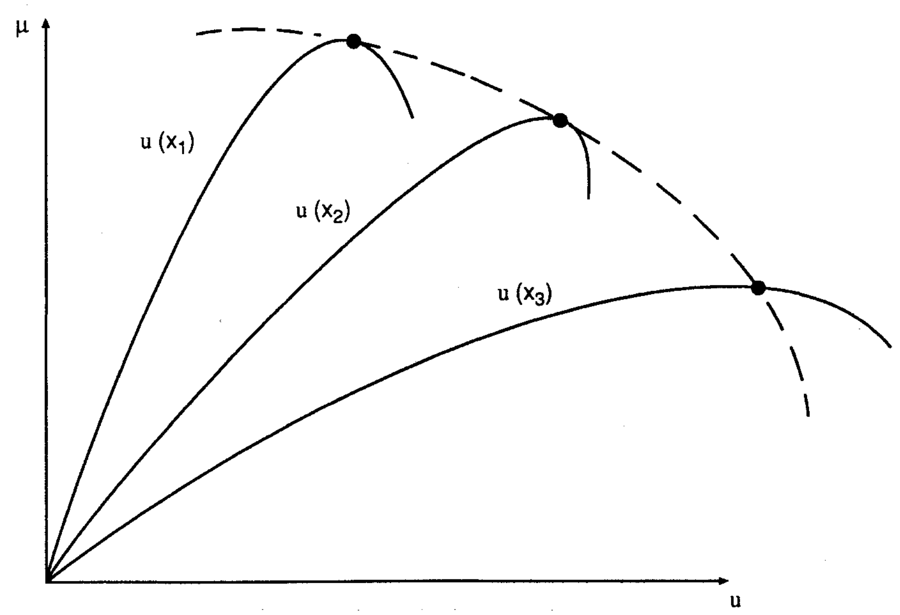

Fig. 1 Limit load equilibrium path.

with J being the Jacobian at an equilibrium point at a load amplitude μ near the limit point. The smallest eigenvalue, $\Delta\mu_1$, of Eq. (14) is an estimate of the difference between the current load amplitude μ and the limit load μ^*. This approach will also yield an estimate of a bifurcation load and not only limit load. However, as reported in the examples that follow and in Ref. 3, the estimate obtained from Eq. (14) is usually quite inaccurate. A more accurate estimate of the limit load may be obtained by following the load path beyond the limit load and then fitting a parabola to three points on the load-displacement curve near the limit point. It is important to select a displacement measure that increases monotonically in that region.

To calculate the sensitivity of the limit load, we need to consider the limit point on two structures corresponding to the design variables taking the values x and $x + \Delta x$, respectively. Both points satisfy the equations of equilibrium, therefore we can consider a path going through the limit points as the structure is changed (see Fig. 1), with the path parameter being x. Differentiating the equations of equilibrium, Eq. (1), along that path, we get

$$J^* \frac{\mathrm{d}u^*}{\mathrm{d}x} + \frac{\partial f}{\partial x} = \frac{\mathrm{d}\mu^*}{\mathrm{d}x} p + \mu^* \frac{\mathrm{d}p}{\mathrm{d}x} \tag{15}$$

where the asterisks denote quantities evaluated at a limit point. To obtain $\mathrm{d}\mu^*/\mathrm{d}x$ from Eq. (15) we need to eliminate the first term by premultiplying it by the left eigenvector of J^* that corresponds to the zero eigenvalue. If the tangential stiffness matrix is not symmetric, then the left eigenvector v, satisfying

$$v^T J^* = 0 \tag{16}$$

is different than the right eigenvector u'. Premultiplying Eq. (15) by v^T and solving for $\mathrm{d}\mu^*/\mathrm{d}x$, we get

$$\frac{\mathrm{d}\mu^*}{\mathrm{d}x} = \frac{v^T[(\partial f/\partial x) - \mu^*(\mathrm{d}p/\mathrm{d}x)]}{v^T p} \tag{17}$$

which is the same as Eq. (23) of Ref. 3, though it is obtained by a different approach. For a limit point the numerator of Eq. (17) is different from zero, with a value of zero associated with a bifurcation point (see, for example, Ref. 6).

The semi-analytical version of Eq. (17)

$$\frac{\mathrm{d}\mu^*}{\mathrm{d}x} \approx \frac{v^T[(\Delta f/\Delta x) - \mu^*(\Delta p/\Delta x)]}{v^T p} \tag{18}$$

is easier to implement in most situations.

Equations (17) and (18) are valid at the limit point. On the other hand, the solution obtained from structural analysis packages will often have points near the limit point but not right at it. If $\partial f/\partial x$ and $\mathrm{d}p/\mathrm{d}x$ are evaluated for μ_a, which is substantially different from μ^*, a problem arises. The error in $\mathrm{d}p/\mathrm{d}x$ is often small or zero, but the error in $\partial f/\partial x$ is substantial (recall that in the linear range this term is proportional to μ). To overcome this effect, it is suggested to assume that $\partial f/\partial x$ is approximately proportional to μ and to replace Eq. (17) by

$$\frac{\mathrm{d}\mu^*}{\mathrm{d}x} \approx \frac{v^T[(\mu^*/\mu_a)(\partial f/\partial x) - \mu^*(\mathrm{d}p/\mathrm{d}x)]}{v^T p} \tag{19}$$

Equation (19) was implemented in the engineering analysis language[7] (EAL) finite element program. The program selects its own set of μ values for climbing up the nonlinear load-displacement curve, using the Riks procedure to continue the analysis beyond the limit point. For the examples that follow, the limit load was estimated by passing a parabola through two points on the stable side of the limit point and one point on the unstable side. Very good accuracy was achieved. The tangential stiffness matrix J is symmetric for these problems, and so the left and right buckling modes are the same, $v = u'$. The buckling mode u' was calculated from Eq. (14) with the geometric stiffness matrix used as an approximation to $\mathrm{d}J/\mathrm{d}\mu$. That mode proved to be satisfactory from the standpoint of estimating $\mathrm{d}\mu/\mathrm{d}x$ from Eq. (19). However, the $\Delta\mu$ obtained from Eq. (14) was typically one or two orders of magnitude too large. When $\mathrm{d}J/\mathrm{d}\mu$ was calculated by finite differences using the two points below the limit points closest to it, the error in $\Delta\mu$ was still about 10–40%. Thus it was concluded that estimating the limit load from the load displacement curve was much more accurate.

Because we do not calculate $\Delta\mu$ from Eq. (14), and only u', we can replace $\mathrm{d}J/\mathrm{d}\mu$ by the unit matrix or any other reasonable, positive definite matrix. The EAL program provides for the solution of two standard eigenvalue problems that are the buckling and vibration problems. A few cases for the examples that follow were done with the vibration mode associated with J as a stiffness matrix instead of the buckling mode. At the limit load the two modes are the same, but when the calculation is based on a point away from the limit point, we can expect a difference. Indeed, for some cases the use of vibration mode instead of buckling mode led to substantial deterioration in accuracy, but for others it led to substantial improvement in accuracy. Since there is no good reason to use the vibration modes, these results are not discussed further.

Logarithmic Derivatives

In the following examples, the derivatives are presented in logarithmic form

$$\frac{\mathrm{d}_l\mu}{\mathrm{d}x} = \frac{\mathrm{d}(\log \mu)}{\mathrm{d}(\log x)} = \frac{x}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x} \tag{20}$$

Logarithmic derivatives not only have the advantage that they are nondimensional and independent of units, but also their magnitudes are easily interpreted. A value of 1 for a logarithmic derivative implies that 1% change in x results in approximately 1% change in μ. It is easy to check that a relationship of the form $\mu = cx^n$ gives a logarithmic derivative of n, no matter what is the value of x. This property provides some immediate checks on the logarithmic derivatives. For example, for a truss structure, if the cross-sectional areas of all of the members are scaled by a factor, the buckling load will increase by the same factor. This means that, if a single design variable controls all of the cross-sectional areas, the logarithmic derivative of the buckling load with respect to this design variable is 1. If, as is more commonly done, several design variables control the cross-sectional areas, then the sum of all of the logarithmic derivatives must add up to 1.

Finally, low values of the logarithmic derivatives usually signal possible accuracy problems for that derivative. If the small derivative reflects the fact that the design variable may not have much influence on the function, then the effect of algorithmic or roundoff errors may overwhelm the weak influence of that variable. If the small derivative merely indicates that we are near a stationary point, then first-order finite difference approximations to the derivatives may have excessive truncation errors.

Truss Dome Example

A 30-member truss dome, analyzed and optimized for minimum weight subject to a constraint on the limit load by Khot and Kamat,[8] Kamat and Ruangsiliasingha,[9] and Wu and Arora,[3] is shown in Fig. 2, with coordinate data given in Table 1. The members are made of aluminum with an elastic modulus $E = 10^7$ psi, and the dome is loaded by a compressive vertical load of 2000 lbs at node 1. The dome cross-sectional areas are grouped in three groups as shown in Table 2, which also gives the initial and final designs reported by Wu and Arora.[3] With the structure loaded by a 2000 load, the initial design yielded a limit load amplitude of 1.0000, based on the

Table 1 Coordinates of the node points of dome structure in inches

Node	x	y	z
1	0.0	0.0	85.912
7	360.0	0.0	64.662
2	180.0	311.769	64.662
19	180.0	0.0	0.0
18	540.0	311.769	21.709
8	360.0	623.538	0.0
9	0.0	623.538	21.709

Table 2 Initial and optimal design for dome truss from Ref. 3

Elements	Initial design area, in.^{2}	Optimal design area, in.^{2}
1, 2, 3, 4, 5, 6	1.5280	1.6108
7, 8, 9, 10, 11, 12	1.5280	1.4627
13, 16, 19, 22, 25, 28	1.5280	0.1
14, 15, 17, 18, 20, 21	—	—
23, 24, 26, 27, 29, 30	—	—

Table 3 Logarithmic derivatives of limit load amplitude with respect to group cross-sectional areas, $(A_i/\mu^*)(\partial\mu^*/\partial A_i)$ for initial and final designs of dome structure

	Group 1		Group 2		Group 3	
μ_a	FD	SA	FD	SA	FD	SA
Initial design						
0.999	0.5539	0.5539	0.4353	0.4330	0.0121	0.0122
0.970	—	0.5599	—	0.4284	—	0.0119
0.897	—	0.5652	—	0.4233	—	0.0115
0.775	—	0.5708	—	0.4180	—	0.0111
Optimal design						
0.992	0.5294	0.5309	0.4802	0.4742	−0.0048	−0.0051
0.882	—	0.5435	—	0.4675	—	−0.0111
0.671	—	0.5556	—	0.4634	—	−0.0190

displacement under the load at $\mu = 0.970$, 0.999, and 0.989. For the final design, the limit load amplitude was predicted as $\mu^* = 0.9997$ from $\mu = 0.882$, 0.992, and 0.973. It is assumed that both designs were limit load critical in Ref. 3, so that these values of μ^* are shown to be very accurate.

Logarithmic derivatives of the limit load with respect to the three group cross-sectional areas for the initial and final designs are shown in Table 3. The semi-analytical derivatives are calculated from several points on the load-displacement curve and compared with overall finite difference derivatives. The semi-analytical derivatives are based on forward differences with 1% step size. Some cases were checked with central differences, but there was almost no difference, indicating that the tangential stiffness matrix is almost linear in the design variables. The overall finite differences are calculated by central differences with a step size of 1%. It was found that forward difference derivatives were a few percent less accurate.

For both initial and optimal designs, it was found that the agreement between the finite difference derivatives and the semi-analytical derivatives calculated at the point closest to the limit point was excellent. An independent check available on the derivatives is that the sum of the three derivatives should be 1. This is due to the fact that if all cross-sectional areas are increased by a factor, the limit load will increase by the same factor. For the initial design the sum of the three finite difference derivatives is 1.0013, and the sum of the semi-analytical derivatives is 0.9991. For the optimal design, the sums are 1.0048 and 1.0000, respectively. This check may indicate that the semi-analytical derivatives are more accurate.

The accuracy of the semi-analytical derivatives with respect to A_1 and A_2 remained good even from points substantially away from the limit point. However, the accuracy of the derivative $\partial\mu^*/\partial A_3$ was poor away from the limit point for the optimal design. However, this loss of accuracy may not be significant because that derivative is so small. Its value indicates that a 1% change in A_3 results only in a 0.005% change in the limit load. The smallness of that derivative is also indicated by its change of sign from the initial to the optimal designs. It is well known that very small derivatives are prone to have large errors in them (e.g., Ref. 2, page 261). Thus, it appears that for this problem satisfactory accuracy can be achieved even if the derivatives of the limit load are calculated based on information substantially below the limit load.

Shallow Cylindrical Shell Example

A hinged shallow cylindrical shell analyzed by Crisfield[10] is used as a second example (see Fig. 3). The nonlinear response was calculated with several sets of initial load increments (which influenced the later load increments selected automatically by the analysis procedure). The estimates of the limit load obtained from three points near the limit point ranged from 583.5 N when the closest point was at $\mu/\mu^* = 0.97$ to 585.7 N when the closest point was at $\mu/\mu^* = 0.9996$.

The logarithmic derivatives of the limit load of the shell with respect to its thickness t and Poisson's ratio ν are given in Table 4. The logarithmic derivative with respect to the thickness is about 2, indicating that the limit load behaves as if it is linear in t^2. The logarithmic derivative with respect to Poisson's ratio is about 0.1, which is consistent with a dependence of the form $1/\sqrt{1-\nu^2}$ [which has a logarithmic derivative of $\nu^2/(1-\nu^2) = 0.099$].

Both sets of derivatives were calculated by central differences with a step size of $\pm 1\%$. Forward difference derivatives were different by about 1% from the central difference derivatives for the thickness and by about 2% for Poisson's ratio (for both semi-analytical and finite difference derivatives).

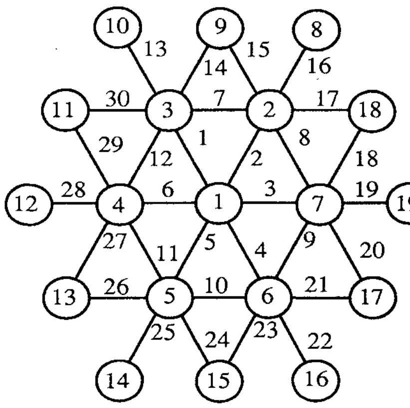

Fig. 2 Plan view of dome structure.

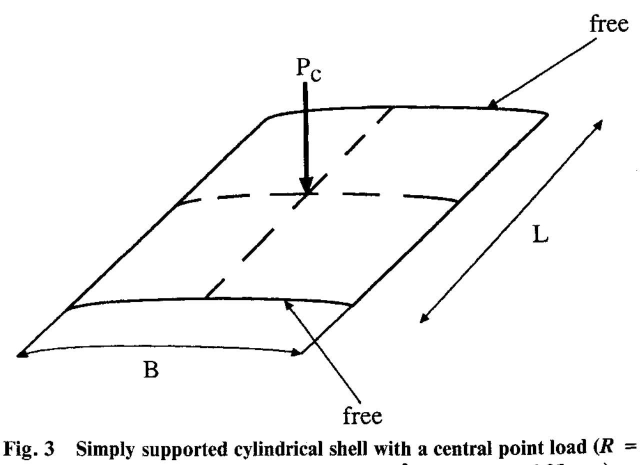

Fig. 3 Simply supported cylindrical shell with a central point load (R = 2540 mm, $L = B = 504$ mm, $E = 3105$ N/mm^2, $\nu = 0.3$, $t = 6.35$ mm).

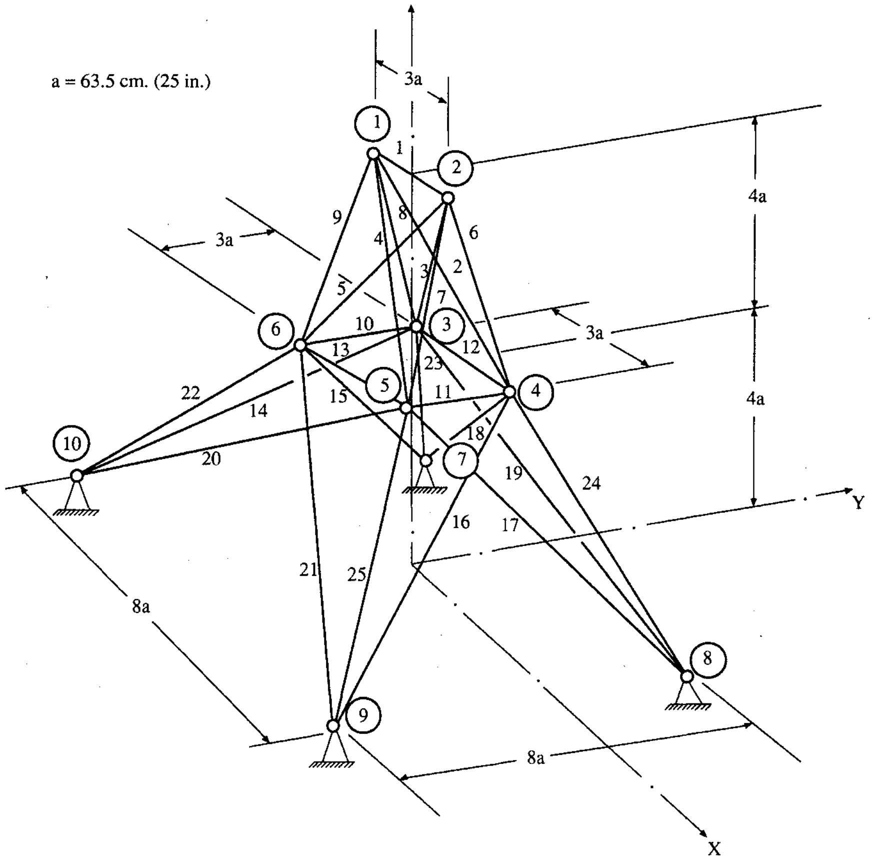

Fig. 4 Twenty-five bar truss.

Table 4 Logarithmic derivatives of limit load amplitude with respect to shell thickness and Poisson's ratio for cylindrical shell

	$\frac{t}{\mu^*}\frac{\partial\mu^*}{\partial t}$		$\frac{\nu}{\mu^*}\frac{\partial\mu^*}{\partial\nu}$	
$\frac{\mu_a}{\mu^*}$	FD	SA	FD	SA
0.9996	2.067	2.073	0.1095	0.1099
0.995	—	2.103	—	0.1136
0.973	—	2.180	—	0.1216
0.928	—	2.294	—	0.1325
0.885	—	2.435	—	0.1454

Table 5 Design data for 25-bar truss

Member areas, in.^{2}			
Design variable	Members	Initial design case NB3, Ref. 3[a]	Optimal design case NB4, Ref.3
1	1	2.0	1.5714
2	2, 3, 4, 5	2.0	3.1195
3	6, 7, 8, 9	2.0	2.1992
4	10, 11, 12, 13	2.0	0.8286
5	14, 15, 16, 17	2.0	3.4330
6	18, 19, 20, 21	2.0	3.7914
7	22, 23, 24, 25	2.0	3.0799
Applied loads, kips			
Node	F_x	F_y	F_z
1	300	−4000	−3000
2	300	4000	−3000

[a]Areas are given as 0.5 in.^{2} in Ref. 3, but it appears to be a typographical error.

Again, there is excellent agreement between the finite difference and semi-analytical derivatives calculated very close to the limit point. The accuracy of the semi-analytical derivatives deteriorates more severely away from the limit point than it did for the truss dome. Again, the small derivative (with respect to Poisson's ratio) deteriorates faster than the large derivative.

Twenty-Five Bar Truss Example

The last example is the well-known 25-bar truss shown in Fig. 4. This case was investigated by Wu and Arora[3] for the design variables and loading shown in Table 5. For the optimum design they have found that accurate analytical derivatives could be obtained only if calculated from points extremely close to the buckling load, and accurate finite difference derivatives required very small step sizes. Herein, logarithmic derivatives were calculated for two of the seven design variables: the first one, which has a small logarithmic derivative, and the sixth one, which has a large logarithmic derivative. Central differences with a 1% step size were used for the finite difference derivatives and were a fraction of a percent more accurate than one-sided differences. For the semi-analytical results, the same step size was used with forward differences, with no noticeable loss of accuracy over central differences (reflecting the fact that the tangential stiffness matrix is almost linear in the design variables). The results are summarized in Table 6. As before, the larger logarithmic derivative is more accurate than the smaller derivative.

Next, the optimal design obtained in Ref. 3, with a critical load factor of 1, was analyzed. The EAL program reports on the number of negative terms on the diagonal of the factored tangential stiffness matrix. When this number changes from zero to one, it indicates that a bifurcation point or a limit point has been encountered. At $\mu = 1.003$, a negative term was first encountered, but the load continued to increase, a second negative term was found at $\mu = 1.004$, and then the load started to decrease. This appears to indicate that a bifurcation point was closely followed by a limit point. This condition is not surprising for an optimal design, since optimization often leads to the coalescence of buckling loads. However, the equations derived in Ref. 3 and here are not applicable to this case

Table 6 Logarithmic derivatives of limit load amplitude with respect to group cross-sectional areas, $(\partial\mu^*/\partial A_i)(A_i/\mu^*)$ for initial design of 25-bar truss

	Group 1			Group 6		
μ_a/μ^*	FD	Ref. 3	SA	FD	Ref. 3	SA
0.9990	0.03035	0.03077	0.03440	0.5077	0.4972	0.5070
0.9979	—	—	0.03524	—	—	0.5090
0.9965	—	—	0.03600	—	—	0.5110
0.9894	—	—	0.03879	—	—	0.5170
0.9640	—	—	0.04326	—	—	0.5233
0.8702	—	—	0.04438	—	—	0.5010

because of the closely spaced critical points and the appearance of a bifurcation point. Thus, the findings of Ref. 3 that good results can be obtained only very close to the buckling load and require extremely small step sizes are to be expected. Indeed, very poor accuracy (an order of magnitude difference) was observed when derivatives were calculated with the semi-analytical method from $\mu = 0.95$. Finite difference calculations indicated that for some perturbations the double nature of the critical point is preserved, whereas for other perturbations the bifurcation point disappears. Since this case is clearly outside the range of validity of the analysis, no further computations of derivatives were undertaken.

Concluding Remarks

Two aspects of the semi-analytical method for sensitivity calculation of nonlinear static response have been explored. The first aspect involves the application of the method to noncritical static response. It was demonstrated that the semi-analytical method is equivalent to the overall finite difference method when only a single Newton iteration is used. The second aspect involves the application of the method to sensitivity calculation of limit loads. A formula for the sensitivity of limit loads based on data at loads lower than the limit loads has been derived. Three examples were used to show that reasonable accuracy can be obtained if the sensitivity is calculated at loads that are a few percent lower than the limit load.

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References

[1]Haftka, R. T., and Adelman, H. M., "Recent Developments in Structural Sensitivity Analysis," *Structural Optimization,* Vol. 1, No. 3, 1989, pp. 137–151.

[2]Haftka, R. T., and Gürdal, Z., *Elements of Structural Optimization,* 3rd ed., Kluwer Publishing Co., Dordrecht, The Netherlands, 1992.

[3]Wu, C. C., and Arora, J. S., "Design Sensitivity Analysis of Non-Linear Buckling Load," *Computational Mechanics,* Vol. 3, 1988, pp. 129–140.

[4]Komarakul-na-nakorn, A., and Arora, J. S., "A Study of Nonlinear Stability Limit and Its Design Sensitivity Analysis," Univ. of Iowa, TR N. ODL-90.10, Iowa City, IA, Oct. 1990.

[5]Barthelemy, B., and Haftka, R. T., "Accuracy Analysis of the Semi-Analytical Method for Shape Sensitivity Calculation," *Mechanics of Structures and Machines,* Vol. 18, No. 3, 1990, pp. 407–432.

[6]Komarakul-na-nakorn, A., and Arora, J. S., "Stability Criteria: A Review," *Computers and Structures,* Vol. 37, No. 1, 1990, pp. 35–49.

[7]Whetstone, J. D., "EISI EAL Engineering Language Reference Manual," Engineering Information Systems, Inc., EISI-EAL System Level 2091, San Jose, CA, July 1983.

[8]Khot, N. S., and Kamat, M. P., "Minimum Weight Design of Truss Structures with Geometric Nonlinear Behavior," *AIAA Journal,* Vol. 23, No. 1, 1985, pp. 139–144.

[9]Kamat, M. P., and Ruangsiliasingha, P., "Design Sensitivity Derivatives in Nonlinear Response," AIAA Paper 84-0973, AIAA /ASME/ASCE/AHS 25th Structures, Structural Dynamics, and Materials Conference, Palm Beach, CA, May 14–16, 1984.

[10]Crisfield, M. A., "A Fast Incremental/Iterative Solution Procedure That Handles 'Snap-Through'," *Computers and Structures,* Vol. 13, 1981, pp. 55–62.